

ON THE RATE OF CONVERGENCE OF FINITE-DIFFERENCE APPROXIMATIONS FOR ELLIPTIC ISAACS EQUATIONS IN SMOOTH DOMAINS

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ABSTRACT. We show that there exists an algebraic rate of convergence of solutions of finite-difference approximations for uniformly elliptic Isaacs in smooth bounded domains.

1. INTRODUCTION

Caffarelli and Souganidis in [5] proved that there is an algebraic rate of convergence of solutions of finite-difference schemes to the Lipschitz continuous viscosity solution of the fully nonlinear elliptic equation

$$F(D^2u(x)) = f(x) \tag{1.1}$$

in a regular domain with Dirichlet boundary data. This is the first result available for fully nonlinear elliptic equations without convexity assumptions on F . Naturally, one would want to extend the result to the F 's depending also on Du , u , and x . Turanova [30] extended the results of [5] to F 's explicitly depending on x . Our main goal here is to show that it is possible by using techniques completely different from the ones in [5] and [30] and based on the observation that solutions with bounded second-order derivatives admit a Hölder estimate of the second order derivatives in \mathcal{L}_d -sense (see Theorem 3.2). However, we are only able to deal with functions F which are written in a minimax form, that is only with Isaacs equations, which do not encompass the most general case of [5] and [30]. The finite-difference approximations we consider are also of more specific type than in [5] and [30]. Our main contribution is doing away with F 's depending only on the second-order derivatives.

The results of this paper bear on elliptic equations. One can obtain similar results for parabolic Isaacs equations (with F even only measurable with respect to the time variable using semi-discretization schemes), because the main technical tool which is a parabolic counterpart of Theorem 3.2 is available owing to Theorem 1.1 of [17]. Then one would “extend” some

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results in [31] to F depending on u, Du , and (t, x) in the same way as the results of the present article “extend” the ones in [5] and [30].

It is natural to compare the results in [5], [30] and in this paper with what is known in the case where F is concave or convex with respect to D^2u . On the one hand, now one is able to treat equations without convexity assumptions. But on the other hand, in the concave or convex case very concrete and rather good estimates are obtained even for degenerate equations with rates, of course, independent of the constant of ellipticity of the equation. In [5], [30] and in this paper the rate is not given in any explicit way and from the proofs one can see that this rate may depend on the constant of ellipticity.

The convergence of and error estimates for monotone and consistent approximations to fully nonlinear, *first-order* PDEs were established a while ago by Crandall and Lions [6] and Souganidis [29].

The convergence of monotone and consistent approximations for fully nonlinear, possibly degenerate second-order PDEs was first proved in Barles and Souganidis [4]. In a series of papers Kuo and Trudinger [22, 23, 24] also looked in great detail at the issues of regularity and existence of such approximations for uniformly elliptic equations.

There is also a probabilistic part of the story related to controlled diffusion processes and Bellman’s equations, which started much earlier, see Kushner [25], Kushner and Dupuis [26], also see Pragarauskas [28].

However, in the above cited articles apart from [6, 29], related to the first-order equations, and [5], [30] no rate of convergence was established. One can read more about the past development of the subject in Barles and Jakobsen [3] and the joint article of Hongjie Dong and the author [8]. Below we concentrate only on some results concerning *second-order* Bellman’s equations, which arise in many areas of mathematics such as control theory, differential geometry, and mathematical finance (see Fleming and Soner [9], Krylov [11]).

The first estimates of the rate of convergence for second-order degenerate or nondegenerate Bellman’s equations appeared in 1997 (see [12]). For equations with constant “coefficients” and arbitrary monotone finite-difference approximations it was proved in [12] that the rate of convergence is $h^{1/3}$ if the error in approximating the true operators with finite-difference ones is of order h on *three* times continuously differentiable functions. The order becomes $h^{1/2}$ if the error in approximating the true operators with finite-difference ones is of order h^2 on *four* times continuously differentiable functions (see Remark 1.4 in [12], which however contains an arithmetical error albeit easily correctable. Also see Theorem 5.1 in [12]). One of the main ideas of [12] is that the equation and its finite-difference approximation should play symmetric roles. The proofs in [12] are purely analytical (in contrast with what one can read in some papers mentioning [12]) even though sometimes probabilistic *interpretation* of some statements are also given. The next step was done in [13] where the so-called method of “shaking

the coefficients” was introduced to deal with the case of *degenerate* parabolic Bellman’s equations with *variable* coefficients. The two sided error estimates were given: from the one side of order $h^{1/21}$ and from the other $h^{1/3}$. Here h (unlike in [12]) was naturally interpreted as the mesh size and the approximating operators were assumed to approximate the true operator with error of order h on *three* times continuously differentiable functions. The order $1/21$ was improved to $1/7$ in [3] in the general setting of [13].

These rates may look unsatisfactory and then one tries to get better estimates on the account of using some special approximations, say providing the error of order h^2 of approximating the main part of the true operator on *four* times continuously differentiable functions. This was already mentioned in Remark 1.4 of [12] and used by Barles and Jakobsen in [1] to extend the results in [12] to equations with variable lower-order “coefficients”.

One can also consider special finite-difference approximations, for instance, only containing pure second-order differences in place of second-order derivatives, when this h^2 approximating error is automatic. In such cases the optimal rate $h^{1/2}$ was obtained in the joint work of Hongjie Dong and the author [8] for degenerate parabolic Bellman’s equations with Lipschitz coefficients in domains. Both ideas of symmetry and “shaking the coefficients” are used in [8] as well as in [7]. In the paper by Hongjie Dong and the author [7] we consider among other things weakly nondegenerate Bellman’s equations with *constant* “coefficients” in the whole space and obtain the rate of convergence h , where h is the mesh size. It may be tempting to say that this result is an improvement of earlier results, however it is just a better rate under different conditions.

It is worth noting that the set of equations satisfying the conditions in [8] is smaller than the one in the papers by Barles and Jakobsen [2, 3], the results of which obtained by using the theory of viscosity solutions guarantee the rate $h^{1/5}$. However, in *the concrete examples* given in Sections 4 in [2, 3] of applications of the general scheme, one gets the rate $h^{1/2}$ according to [8] if one adds the requirement that the coefficients be twice differentiable (and in the case of diagonally dominant matrices replaces Kushner’s approximation with a different one avoiding using mixed finite differences, which is possible as shown in [14]). Moreover, if the equation is uniformly nondegenerate, then the rate is at least $h^{2/3}$ in the elliptic case according to the result of [19]. In [2, 3] the coefficients are only assumed to be once differentiable and still the rate $h^{1/5}$ is guaranteed regardless of degeneracy or nondegeneracy. One more point to be noted is that in [3] parabolic equations are considered with various types of approximation such as Crank-Nicholson and splitting-up schemes related to the time derivative combined with not necessarily finite-difference approximation of differential operators.

In conclusion of the introduction we fix some notation. Let $\mathbb{R}^d = \{x = (x_1, \dots, x_d)\}$ be a d -dimensional Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and let \mathbb{S} be the set of $d \times d$ symmetric matrices. For a fixed constant

$\delta \in (0, 1]$ denote by \mathbb{S}_δ the subset of \mathbb{S} consisting of matrices with eigenvalues in $[\delta, \delta^{-1}]$. For a function $F(u, x)$ given for

$$u = (u', u''), \quad u' = (u'_0, \dots, u'_d) \in \mathbb{R}^{d+1}, \quad u'' \in \mathbb{S}, \quad x \in \mathbb{R}^d$$

and smooth real-valued function $u(x)$ on \mathbb{R}^d we set $D_i u = \partial u / \partial x_i$, $D_{ij} u = D_i D_j u$,

$$F[u](x) = F(u(x), Du(x), D^2 u(x), x),$$

where $Du = (D_i u)$ is the gradient of u and $D^2 u = (D_{ij} u)$ is its Hessian.

Let G be a fixed open bounded subset of \mathbb{R}^d with C^2 boundary.

2. MAIN RESULT

Let $d_1 \geq d$ be an integer. Assume that we are given separable metric spaces A and B and let, for each $\alpha \in A$ and $\beta \in B$, the following functions on \mathbb{R}^d be given:

- (i) $d \times d_1$ matrix-valued $\sigma^{\alpha\beta}(x) = (\sigma_{ij}^{\alpha\beta}(x))$,
- (ii) \mathbb{R}^d -valued $b^{\alpha\beta}(x) = (b_i^{\alpha\beta}(x))$, and
- (iii) real-valued functions $c^{\alpha\beta}(x)$, $f^{\alpha\beta}(x)$.

Assumption 2.1. (i) All the above functions are continuous with respect to $\beta \in B$ for each (α, x) and continuous with respect to $\alpha \in A$ uniformly with respect to $\beta \in B$ for each x . These functions are Borel measurable functions of (α, β, x) , and $c^{\alpha\beta} \geq 0$.

- (ii) For any $x, y \in \mathbb{R}^d$ and $(\alpha, \beta) \in A \times B$

$$\|\sigma^{\alpha\beta}(x)\|, |b^{\alpha\beta}(x)|, |c^{\alpha\beta}(x)|, |f^{\alpha\beta}(x)| \leq \delta^{-1},$$

$$\|\sigma^{\alpha\beta}(x) - \sigma^{\alpha\beta}(y)\| + |b^{\alpha\beta}(x) - b^{\alpha\beta}(y)| \leq \delta^{-1}|x - y|,$$

where for a matrix σ we denote $\|\sigma\|^2 = \text{tr } \sigma \sigma^*$.

- (iii) There exists a constant $\gamma_1 \in (0, 1]$ such that for any $x, y \in \mathbb{R}^d$ and $(\alpha, \beta) \in A \times B$ we have

$$|c^{\alpha\beta}(x) - c^{\alpha\beta}(y)| + |f^{\alpha\beta}(x) - f^{\alpha\beta}(y)| \leq \delta^{-1}|x - y|^{\gamma_1}.$$

- (iv) For $\alpha \in A$, $\beta \in B$, and $x \in \mathbb{R}^d$ we have $a^{\alpha\beta}(x) \in \mathbb{S}_\delta$, where $a^{\alpha\beta} = (a_{ij}^{\alpha\beta}) = (1/2)\sigma^{\alpha\beta}(\sigma^{\alpha\beta})^*$.

We are going to deal with

$$H(u, x) := \sup_{\alpha \in A} \inf_{\beta \in B} [a_{ij}^{\alpha\beta}(x) u''_{ij} + b_i^{\alpha\beta}(x) u'_i - c^{\alpha\beta}(x) u'_0 + f^{\alpha\beta}(x)]$$

(the summation convention over repeated indices is enforced and here the summations are done before other operations are performed), which is a typical object in the theory of stochastic differential games. As in [21] one associates with the above objects and zero boundary data on ∂G a value function $v(x)$. According to [21] the dynamic programming principle holds, which along with the continuity of the data allows us to use the results of [10] and conclude that v is a unique viscosity solution of class $C(\bar{G})$ of $H[v] = 0$ in G with zero boundary data.

Next, for smooth functions $u(x)$ and $\alpha \in A$ and $\beta \in B$ introduce

$$L^{\alpha\beta}u(x) = a_{ij}^{\alpha\beta}(x)D_{ij}u(x) + b_i^{\alpha\beta}(x)D_iu(x) - c^{\alpha\beta}(x)u(x).$$

As is well known (see, for instance, [23]), there exists a finite set $\Lambda = \{l_1, \dots, l_{d_2}\} \subset \mathbb{Z}^d$ containing all vectors from the standard orthonormal basis of \mathbb{R}^d such that one has the following representation

$$L^{\alpha\beta}u(x) = a_k^{\alpha\beta}(x)D_{l_k}^2u(x) + \bar{b}_k^{\alpha\beta}(x)D_{l_k}u(x) - c^{\alpha\beta}(x)u(x),$$

where $D_{l_k}u(x) = \langle Du, l_k \rangle$, $a_k^{\alpha\beta}$ and $\bar{b}_k^{\alpha\beta}$ are certain bounded functions and $a_k^{\alpha\beta} \geq \delta_1$, with a constant $\delta_1 > 0$. One can even arrange for such representation to have the coefficients $a_k^{\alpha\beta}$ and $\bar{b}_k^{\alpha\beta}$ with the same regularity properties with respect to x as the original ones $a_{ij}^{\alpha\beta}$ and $b_i^{\alpha\beta}$ (see, for instance, Theorem 3.1 in [15]). Define B as the smallest closed ball containing Λ , and for $h > 0$ set $\mathbb{Z}_h^d = h\mathbb{Z}^d$,

$$G_h = G \cap \mathbb{Z}_h^d, \quad G_h^o = \{x \in \mathbb{Z}_h^d : x + hB \in G\}, \quad \partial_h G = G_h \setminus G_h^o.$$

Next, for $h > 0$ we introduce

$$\begin{aligned} \delta_{h,l_k}u(x) &= \frac{u(x + hl_k) - u(x)}{h}, \\ \Delta_{h,l_k}u(x) &= \frac{u(x + hl_k) - 2u(x) + u(x - hl_k)}{h^2}, \\ L_h^{\alpha\beta}u(x) &= a_k^{\alpha\beta}(x)\Delta_{h,l_k}u(x) + \bar{b}_k^{\alpha\beta}(x)\delta_{h,l_k}u(x) - c^{\alpha\beta}(x)u(x), \\ H_h[u](x) &= \sup_{\alpha \in A} \inf_{\beta \in B} [L_h^{\alpha\beta}u(x) + f^{\alpha\beta}(x)]. \end{aligned}$$

It is a simple fact shown, for instance, in [23] that for each sufficiently small h there exists a unique function v_h on G_h such that $H_h[v_h] = 0$ on G_h^o and $v_h = 0$ on $\partial_h G$.

Here is our main result:

Theorem 2.1. *There exist constants N and $\beta > 0$ such that for all sufficiently small $h > 0$ we have on G_h that*

$$|v_h - v| \leq Nh^\beta.$$

The proof of Theorem 2.1 is given in Section 5 after some preparations are made.

3. A GENERAL SCHEME

Denote by \mathbb{L}_δ the set of differential operators of the form

$$Lu(x) = a_{ij}(x)D_{ij}u(x) + b_i(x)D_iu(x) + c(x)u(x),$$

where $a(x) = (a_{ij}(x))$, $b(x) = (b_i(x))$, and $c(x)$ are \mathbb{S}_δ -valued, \mathbb{R}^d -valued, and real-valued Borel functions, respectively, on \mathbb{R}^d satisfying

$$|b| \leq \delta^{-1}, \quad 0 \leq -c \leq \delta^{-1}.$$

Here is Corollary 3.5 of [18] which slightly generalizes the main result in Fang-Hua Lin [27].

Lemma 3.1. *There exist $\gamma_0 \in (0, 1)$ and N depending only on δ , d , and G such that for any $L \in \mathbb{L}_\delta$, $\gamma \in (0, \gamma_0]$, and $u \in W_{d,loc}^2(G) \cap C(\bar{G})$ we have*

$$\int_G (|D^2 u|^\gamma + |Du|^\gamma) dx \leq N \|Lu\|_{\mathcal{L}_d(G)}^\gamma + N \sup_{\partial G} |u|^\gamma. \quad (3.1)$$

Next, we consider a function $F(u, x)$ given for

$$u = (u', u''), \quad u' = (u'_0, \dots, u'_d) \in \mathbb{R}^{d+1}, \quad u'' \in \mathbb{S}, \quad x \in \mathbb{R}^d.$$

Assumption 3.1. (i) The function $F(u, x)$ is Lipschitz continuous with Lipschitz constant δ^{-1} with respect to u ,

(ii) For any x at all point of differentiability of $F(u, x)$ with respect to u , we have $F_{u''} \in \mathbb{S}_\delta$, $F_{u'_0} \leq 0$,

(iii) There is a constant $\gamma_1 \in (0, 1]$ such that for any $x, y \in \mathbb{R}^d$ and $u = (u', u'')$ such that $|x - y| \leq 1$ we have

$$|F(u, x) - F(u, y)| \leq \delta^{-1} |x - y|^{\gamma_1} (1 + |u|).$$

For $\varepsilon > 0$ introduce

$$G^\varepsilon = \{x : \exists y \in G, |y - x| \leq \varepsilon\}, \quad B_\varepsilon = \{x : |x| < \varepsilon\}.$$

The following result is one of our main technical tools.

Theorem 3.2. *Let a function $u \in W_\infty^2(G^\varepsilon)$ satisfy $F[u] = 0$ in G^ε , and let $p \in (\gamma_0, \infty)$. Then for any $h \in [0, \varepsilon \wedge 1]$ and $|l| = 1$*

$$\|u(hl + \cdot) - u\|_{W_p^2(G)} \leq Nh^{\gamma_1 \gamma_0 / p} (1 + M_\varepsilon),$$

where

$$M_\varepsilon := \|u\|_{W_\infty^2(G^\varepsilon)}$$

and the constant N depends only on G , δ , p , and d .

Proof. Since the increments of the function and its first derivatives are well controlled by the product of Nh times M_ε , it suffices to prove that

$$\|D^2 u(hl + \cdot) - D^2 u\|_{\mathcal{L}_p(G)} \leq Nh^{\gamma_1 \gamma_0 / p} (1 + M_\varepsilon). \quad (3.2)$$

Denote $v(x) = u(hl + x) - u(x)$ and observe that v satisfies

$$Lv + f = 0$$

in G , where $L \in \mathbb{L}_\delta$ and $f(x) = F(u(x), Du(x), D^2 u(x), x + hl) - F[u](x)$, so that

$$|f| \leq \delta^{-1} h^{\gamma_1} (1 + M_\varepsilon).$$

Owing to Lemma 3.1 for any $\lambda > 0$ we have

$$|G \cap \{|D^2 v|^p \geq \lambda\}| \leq \frac{1}{\lambda^{\gamma_0/p}} \int_G |D^2 v|^{\gamma_0} dx \leq N \frac{h^{\gamma_1 \gamma_0}}{\lambda^{\gamma_0/p}} (1 + M_\varepsilon^{\gamma_0}).$$

We also note that if $\lambda > (2M_\varepsilon)^p$, then $G \cap \{|D^2 v|^p \geq \lambda\} = \emptyset$. It follows that

$$\begin{aligned} \|D^2 u(h + \cdot) - D^2 u\|_{\mathcal{L}_p(G)}^p &\leq N(1 + M_\varepsilon^{\gamma_0}) h^{\gamma_1 \gamma_0} \int_0^{(2M_\varepsilon)^p} \lambda^{-\gamma_0/p} d\lambda \\ &\leq N h^{\gamma_1 \gamma_0} (1 + M_\varepsilon^p) \end{aligned}$$

and the theorem is proved.

We say that a number $\varepsilon_0 > 0$ is sufficiently small if Lemma 3.1 holds with the same γ_0 , a constant N which is twice the constant from (3.1), and with G^{ε_0} in place of G . The fact that the set of sufficiently small ε_0 contains $[0, \alpha)$ with $\alpha > 0$ follows from the way Corollary 3.5 of [18] is proved and from the fact that the boundaries of G^ε have the same regularity as that of G if ε is small enough.

Take a nonnegative symmetric $\zeta \in C_0^\infty(B_1)$ with unit integral, for $\varepsilon > 0$ define $\zeta_\varepsilon(x) = \varepsilon^{-d} \zeta(x/\varepsilon)$, and for functions v on \mathbb{R}^d set $v^{(\varepsilon)} = \zeta_\varepsilon * v$.

Lemma 3.3. *Let $\varepsilon_0 > 0$ be sufficiently small, $p \in [1, \infty)$, and $u \in W_\infty^2(G^{\varepsilon_0})$. Assume that $F[u] = 0$ in G^{ε_0} . Then for any $h, \varepsilon \in (0, \varepsilon_0/4]$ there exists an x_0 with $|x_0| \leq h$ for which*

$$\begin{aligned} \sum_{x \in G_h} |F(u^{(\varepsilon)}(x + x_0), Du^{(\varepsilon)}(x + x_0), D^2 u^{(\varepsilon)}(x + x_0), x)|^p h^d \\ \leq N(h^{p\gamma_1} + \varepsilon^{\gamma_1 \gamma_0})(1 + M_{\varepsilon_0}^p), \end{aligned} \quad (3.3)$$

where N depends only on G , δ , p , and d .

Proof. The terms in the sum in (3.3) are obviously less than h^d times $2^p I(x + x_0) + 2^p J(x, x_0)$, where

$$\begin{aligned} I(x) &= |F[u^{(\varepsilon)}](x) - F[u](x)|^p, \\ J(x, x_0) &= |F(u^{(\varepsilon)}(x + x_0), Du^{(\varepsilon)}(x + x_0), D^2 u^{(\varepsilon)}(x + x_0), x) \\ &\quad - F(u^{(\varepsilon)}(x + x_0), Du^{(\varepsilon)}(x + x_0), D^2 u^{(\varepsilon)}(x + x_0), x + x_0)|^p. \end{aligned}$$

By assumption $J(x, x_0) \leq N h^{p\gamma_1} (1 + M_{\varepsilon_0}^p)$, so that

$$\sum_{x \in G_h} J(x, x_0) h^d \leq N h^{p\gamma_1} (1 + M_{\varepsilon_0}^p)$$

whenever $|x_0| \leq h$.

Next, observe that

$$\sum_{x \in G_h} I_{x+B_{h/2}} \leq I_{G^{\varepsilon_0/2}}$$

implying that

$$\begin{aligned} \int_{B_{h/2}} \sum_{x \in G_h} I(x + x_0) h^d dx_0 &= N \sum_{x \in G_h} \int_{B_{h/2}} I(x + x_0) dx_0 \\ &= \sum_{x \in G_h} \int_{x+B_{h/2}} I(y) dy \leq \int_{G^{\varepsilon_0/2}} I(x) dx. \end{aligned}$$

Furthermore, for $x \in G^{\varepsilon_0/2}$ we have

$$|D^2 u^{(\varepsilon)}(x) - D^2 u(x)|^p \leq N \int_{B_\varepsilon} |D^2 u(x+y) - D^2 u(x)|^p dy.$$

Similar relations are true for the first order derivatives and functions themselves. Therefore, in light of Theorem 3.2

$$\int_{G^{\varepsilon_0/2}} I(x) dx \leq N \int_{B_\varepsilon} \|u(\cdot + y) - u\|_{W_p^2(G^{\varepsilon_0/2})}^p dy \leq N_1 \varepsilon^{\gamma_1 \gamma_0} (1 + M_{\varepsilon_0}^p).$$

We conclude that there exists an x_0 with $|x_0| < h/2$ such that

$$\sum_{x \in G_h} I(x + x_0) h^d \leq 2N_1 \varepsilon^{\gamma_1 \gamma_0} (1 + M_{\varepsilon_0}^p).$$

This along with the above estimate of J proves the lemma.

Below in the paper Λ is not necessarily the set from Section 2 and we introduce G^o and $\partial_h G$ in the same way as in Section 2 for any Λ we choose.

Definition 3.1. Let $\Lambda \subset \mathbb{Z}^d$. We say that an operator H_h defined on functions on \mathbb{Z}_h^d and mapping them into functions on \mathbb{Z}_h^d is Λ -local if for any $x_0 \in \mathbb{Z}_h^d$ and functions u and v on \mathbb{Z}_h^d such that $u = v$ on $x_0 + h\Lambda$ we have $H_h[u](x_0) = H_h[v](x_0)$.

Definition 3.2. Let $\mathcal{H} = \{H_h : h \in (0, 1)\}$ be a family of Λ -local operators H_h mapping functions on \mathbb{Z}_h^d to functions on \mathbb{Z}_h^d and $p \geq 1$. We say that the family is $\Lambda \mathcal{L}_p$ -stable in G if there exists constants $N = N(\mathcal{H}, p)$ and $h_0 = h_0(\mathcal{H}, p) > 0$ such that, for any $h \in (0, h_0]$ and functions v and u on \mathbb{Z}_h^d , in G_h we have

$$u - v \leq N \left(\sum_{x \in G_h^o} (H_h[u](x) - H_h[v](x))_-^p h^d \right)^{1/p} + \max_{\partial_h G} (u - v)_+.$$

Lemma 3.4. Assume that \mathcal{H} is $\Lambda \mathcal{L}_p$ -stable in G . Also assume that

$$F[u] - H_h[u] \geq -N_1 h^{\gamma_1} \|u\|_{C^3(G)}, \quad (3.4)$$

in G_h^o for any $u \in C^3(\bar{G})$ and $h \in (0, 1)$ or

$$F[u] - H_h[u] \leq N_1 h^{\gamma_1} \|u\|_{C^3(G)}, \quad (3.5)$$

in G_h^o for any $u \in C^3(\bar{G})$ and $h \in (0, h_0]$, where N_1 is a constant independent of u and h . Finally, suppose that for each $h \in (0, h_0]$ we are given a function v_h on G_h such that $H_h[v_h] = 0$ on G_h^o . Then for any sufficiently small $\varepsilon_0 > 0$ and $h \leq h_0$ satisfying $h^{\gamma_1/2} \leq \varepsilon_0/4$ and $u \in W_\infty^2(G^{\varepsilon_0})$ such that $F[u] = 0$ in G^{ε_0} we have that on G_h

$$v_h \leq u + \max_{\partial_h G} (v_h - u)_+ + Nh^\alpha (1 + M_{\varepsilon_0}) \quad (3.6)$$

if (3.4) holds and

$$u \leq v_h + \max_{\partial_h G} (u - v_h)_+ + Nh^\alpha (1 + M_{\varepsilon_0}) \quad (3.7)$$

if (3.5) holds, where N and $\alpha \in (0, \gamma_1/2)$ depend only on N_1 , $N(\mathcal{H}, p)$, G , δ , p , and d .

Proof. First assume that (3.4) holds. By Lemma 3.3 for any $h, \varepsilon \in (0, \varepsilon_0/4]$ there exists an x_0 with $|x_0| \leq h$ for which (3.3) holds. Owing to (3.4) and the fact that $H_h[v_h] = 0$, if $x \in G_h^o$, we infer that

$$\begin{aligned} & \sum_{x \in G_h^o} [H_h(u^{(\varepsilon)}(\cdot + x_0))(x) - H_h[v_h](x)]_+^p h^d \\ & \leq N(h^{p\gamma_1} + \varepsilon^{\gamma_1\gamma_0})(1 + M_{\varepsilon_0}^p) + NN_1^p h^{p\gamma_1} \|u^{(\varepsilon)}\|_{C^3(G^{\varepsilon_0/4})}^p. \end{aligned}$$

According to Definition 3.2, if $h \leq h_0$, in G_h

$$\begin{aligned} v_h & \leq u^{(\varepsilon)}(\cdot + x_0) + N(h^{\gamma_1} + \varepsilon^{\gamma_1\gamma_0/p})(1 + M_{\varepsilon_0}) + NN_1 h^{\gamma_1} \|u^{(\varepsilon)}\|_{C^3(G^{\varepsilon_0/4})} \\ & \quad + \max_{\partial_h G} (v_h - u^{(\varepsilon)}(\cdot + x_0))_+. \end{aligned}$$

One knows that

$$\|u^{(\varepsilon)}\|_{C^3(G^{\varepsilon_0/4})} \leq N\varepsilon^{-1}M_{\varepsilon_0}.$$

Furthermore, for $x \in G$

$$u^{(\varepsilon)}(x + x_0) \geq u(x + x_0) - NM_{\varepsilon_0}\varepsilon^2 \geq u(x) - NM_{\varepsilon_0}(h + \varepsilon^2)$$

and $u^{(\varepsilon)}(x + x_0)$ admits a similar estimate from above.

Hence, in G_h for any $h, \varepsilon \in (0, \varepsilon_0/4]$ satisfying $h \leq h_0$ we have

$$\begin{aligned} v_h & \leq u + \max_{\partial_h G} (v_h - u)_+ \\ & \quad + NM_{\varepsilon_0}(h + \varepsilon^2) + N(h^{\gamma_1} + \varepsilon^{\gamma_1\gamma_0/p})(1 + M_{\varepsilon_0}) + NN_1 h^{\gamma_1} \varepsilon^{-1} M_{\varepsilon_0} \end{aligned}$$

and the result follows if we take $\varepsilon = h^{\gamma_1/2}$ and restrict h to satisfy $h^{\gamma_1/2} \leq \varepsilon_0/4$ in order to have $\varepsilon \leq \varepsilon_0/4$.

This proves our assertion concerning (3.6). The remaining assertion is proved similarly. The lemma is proved.

4. APPLICATION OF THE GENERAL SCHEME

Let $H(u, x)$ be a function satisfying Assumption 3.1 and such that

$$|H(0, x)| \leq \delta^{-1}. \quad (4.1)$$

We start with the following consequences of Theorems 1.1 and 1.3 of [16].

Theorem 4.1. *There is a constant $\hat{\delta} \in (0, \delta]$ depending only on δ and d and there exists a function $P(u)$ (independent of x), satisfying Assumption 3.1 with $\hat{\delta}$ in place of δ , such that for any $K \geq 0$ and sufficiently small $\varepsilon_0 \geq 0$ each of the equations*

$$\max(H[v], P[v] - K) = 0 \quad (4.2)$$

and

$$\min(H[v], -P[-v] + K) = 0 \quad (4.3)$$

in G^{ε_0} (a.e.) with boundary condition $v = 0$ on $\partial G^{\varepsilon_0}$ has a unique solution in the space $C^{0,1}(\bar{G}^{\varepsilon_0}) \cap C_{loc}^{1,1}(G^{\varepsilon_0})$. Denote by $v_{\varepsilon_0, K}$ the solution of (4.2) and by $v_{\varepsilon_0, -K}$ the solution of (4.3). Then

$$|v_{\varepsilon_0, \pm K}|, |Dv_{\varepsilon_0, \pm K}|, \rho_{\varepsilon_0} |D^2 v_{\varepsilon_0, \pm K}| \leq N(1 + K) \quad \text{in } G^{\varepsilon_0} \quad (\text{a.e.}), \quad (4.4)$$

where

$$\rho_{\varepsilon_0}(x) = \text{dist}(x, \mathbb{R}^d \setminus G^{\varepsilon_0}),$$

and N is a constant depending only on G and δ .

Finally, $P(u)$ is constructed on the sole basis of δ and d , it is positive homogeneous of degree one and convex in u .

Actually, Theorems 1.1 and 1.3 of [16] are proved only for $\varepsilon_0 = 0$. The fact that they also hold for sufficiently small $\varepsilon_0 > 0$ easily follows by inspecting their proofs.

Theorem 4.2. Assume that we are given a $\Lambda\mathcal{L}_p$ -stable family \mathcal{H} of operators H_h . Also suppose that

$$H[u] - H_h[u] \geq -N_1 h^{\gamma_1} \|u\|_{C^3(G)}, \quad (4.5)$$

in G_h^o for any $u \in C^3(\bar{G})$ and $h \in (0, h_0]$ or

$$H[u] - H_h[u] \leq N_1 h^{\gamma_1} \|u\|_{C^3(G)}, \quad (4.6)$$

in G_h^o for any $u \in C^3(\bar{G})$ and $h \in (0, h_0]$, where N_1 is a constant independent of u and h . Finally, suppose that for each $h \in (0, h_0]$ we are given a function v_h on G_h such that $H_h[v_h] = 0$ on G_h^o and $v_h = 0$ on $\partial_h G$. Then there exists an $h_1 \in (0, h_0]$ depending only on G , h_0 , α and δ such that for any $K \geq 0$ and $h \in (0, h_1]$ on G_h we have

$$v_h \leq v_{0, K} + N(1 + K)h^{\alpha/2}, \quad (4.7)$$

if (4.5) holds and

$$v_h \geq v_{0, -K} + N(1 + K)h^{\alpha/2} \quad (4.8)$$

in case (4.6) holds, where N depends only on Λ , N_1 , $N(\mathcal{H}, p)$, G , δ , p , and d and α is the constant from Lemma 3.4.

Proof. First suppose that (4.5) holds. For K fixed introduce $F(u, x) = \max(H(u, x), P(u) - K)$ and observe that $F \geq H$, so that (4.5) implies (3.4). Therefore, we can apply Lemma 3.4 with $v_{\varepsilon_0, K}$ for $\varepsilon_0 > 0$ in place of u .

Notice that $\rho_{\varepsilon_0} \geq \varepsilon_0$ on G , so that $M_{\varepsilon_0} \leq N(1 + K)\varepsilon_0^{-1}$ by Theorem 4.1. By the same theorem $|v_{\varepsilon_0, K}| \leq N\varepsilon_0(1 + K)$ in $\partial_h G$ if $h \leq \varepsilon_0$. Hence, for h also satisfying $h \leq h_0$ and $4h^{\gamma_1/2} \leq \varepsilon_0$ and sufficiently small ε_0 , on G_h

$$v_h \leq v_{\varepsilon_0, K} + N\varepsilon_0(1 + K) + Nh^\alpha(1 + \varepsilon_0^{-1}(1 + K)). \quad (4.9)$$

Next, for a constant N depending only on G and δ we have that $v_{\varepsilon_0, K} \leq N\varepsilon_0(1 + K) = v_{0, K} + N\varepsilon_0(1 + K)$ on ∂G . By the maximum principle this inequality extends to G and in light of (4.9) yields that on G_h

$$v_h \leq v_{0, K} + N\varepsilon_0(1 + K) + Nh^\alpha(1 + \varepsilon_0^{-1}(1 + K)). \quad (4.10)$$

We take here $\varepsilon_0 = 4h^{\alpha/2}$ and call h_1 the largest value of $h \in (0, h_0]$ for which $\varepsilon_0 = 4h_1^{\alpha/2}$ is sufficiently small and less than 1. Observe that then $4h^{\gamma_1/2} \leq \varepsilon_0$ for $h \leq h_1$ because $\alpha \leq \gamma_1$ and $h_1 \leq 1$. Now (4.10) yields (4.7).

The assertion about (4.8) is proved similarly. The theorem is proved.

Corollary 4.3. *Assume that we are given a $\Lambda\mathcal{L}_p$ -stable family \mathcal{H} of operators H_h . Also suppose that*

$$|H[u] - H_h[u]| \leq N_1 h^{\gamma_1} \|u\|_{C^3(G)}, \quad (4.11)$$

in G_h^o for any $u \in C^3(\bar{G})$ and $h \in (0, h_0]$, where N_1 is a constant independent of u and h . Suppose that for each $h \in (0, h_0]$ we are given a function v_h on G_h such that $H_h[v_h] = 0$ on G_h^o and $v_h = 0$ on $\partial_h G$. Finally, suppose that there exist constants N_2 and $\gamma_2 > 0$ such that for the unique viscosity solution $v \in C(\bar{G})$ of equation $H[v] = 0$ in G with zero boundary data we have

$$v_{0,-K} + N_2 K^{-\gamma_2} \geq v \geq v_{0,K} - N_2 K^{-\gamma_2} \quad (4.12)$$

in G for any $K \geq 1$. Then there exist constants $\beta \in (0, 1)$ and N such that for all sufficiently small $h > 0$ we have that on G_h

$$|v_h - v| \leq N h^\beta.$$

Indeed, we have for $K \geq 1$ and $h \in (0, h_1]$ that on G_h

$$v_h \leq v_{0,K} + N(1+K)h^{\alpha/2} \leq v + N(1+K)h^{\alpha/2} + N_2 K^{-\gamma_2}$$

and to show that $v_h \leq v + N h^\beta$, it only remains to minimize the last expression with respect to $K \geq 1$. The estimate from the other side is obtained similarly.

5. PROOF OF THEOREM 2.1

Obviously, H is Lipschitz continuous with respect to u with Lipschitz constant depending only on δ and d . This function is also decreasing with respect to u'_0 and therefore $F_{u'_0} \leq 0$ wherever $F_{u'_0}$ exists. Furthermore, for $t > 0$ and $x, \lambda \in \mathbb{R}^d$ we have

$$H(u', u'' + t\lambda\lambda^*, x) \geq H(u, x) + t\delta|\lambda|^2,$$

which shows that $H_{u''}$, whenever it exists, is uniformly nondegenerate, so that Assumption 3.1 (ii) is satisfied with a perhaps different $\delta > 0$. Assumption 3.1 (iii) is satisfied as well with a perhaps different $\delta > 0$.

The properties of P listed in Theorem 4.1 or just the construction of P in [16] yield that there is a set A_2 and bounded continuous functions $\sigma^\alpha = \sigma^{\alpha\beta}$, $b^\alpha = b^{\alpha\beta}$, $c^\alpha = c^{\alpha\beta}$ (independent of x and β), and $f^{\alpha\beta} \equiv 0$ defined on A_2 such that Assumption 2.1 is satisfied perhaps with a different constant $\delta > 0$ and for $a^\alpha := a^{\alpha\beta} = (1/2)\sigma^\alpha(\sigma^\alpha)^*$ we have

$$P[u](x) = \sup_{\alpha \in A_2} [a_{ij}^\alpha D_{ij}u(x) + b_i^\alpha D_i u(x) - c^\alpha u(x)]. \quad (5.1)$$

Define $A_1 = A$,

$$\hat{A} = A_1 \cup A_2$$

and observe that

$$\begin{aligned} & \max(H[u](x), P[u](x) - K) \\ &= \max \left\{ \sup_{\alpha \in A_1} \inf_{\beta \in B} [L^{\alpha\beta} u(x) + f^{\alpha\beta}(x)], \sup_{\alpha \in A_2} \inf_{\beta \in B} [L^{\alpha\beta} u(x) + f^{\alpha\beta}(x) - K] \right\} \\ &= \sup_{\alpha \in \hat{A}} \inf_{\beta \in B} [L^{\alpha\beta} u(x) + f_K^{\alpha\beta}(x)] \quad (f_K^{\alpha\beta}(x) = f^{\alpha\beta}(x)I_{\alpha \in A_1} - KI_{\alpha \in A_2}), \end{aligned}$$

where the first equality follows from the definition of $H[u]$, (5.1), and the fact that $L^{\alpha\beta}$ is independent of β for $\alpha \in A_2$.

We have just repeated part of the proof of Theorem 2.3 of [21]. Then, as there, we have a probabilistic representation of $v_{0,K}$, which allows us to conclude that $|v_{0,K} - v| \leq N_2 K^{-\gamma_2}$ in G by inspecting the proof of Theorem 5.2 of [21] in which the convergence $v_{0,K} \rightarrow v$ is proved under much weaker assumptions on $c^{\alpha\beta}$ and $f^{\alpha\beta}$. Exploiting the Hölder continuity of $c^{\alpha\beta}$ and $f^{\alpha\beta}$ easily yields the stated rate of convergence of $v_{0,K}$ to v .

Passing to $v_{0,-K}$ denote $B_1 = B$, $B_2 = A_2$,

$$\hat{B} = B_1 \cup B_2$$

and notice that

$$-P[-u](x) = \inf_{\beta \in B_2} [a_{ij}^\beta D_{ij}u(x) + b_i^\beta D_i u(x) - c^\beta u(x)].$$

Next, as in [21]

$$\begin{aligned} & \min(H[u](x), -P[-u](x) + K) \\ &= \sup_{\alpha \in A} \min \left\{ \inf_{\beta \in B_1} [L^{\alpha\beta} u(x) + f^{\alpha\beta}(x)], \inf_{\beta \in B_2} [L^{\alpha\beta} u(x) + f^{\alpha\beta}(x) + K] \right\} \\ &= \sup_{\alpha \in A} \inf_{\beta \in \hat{B}} [L^{\alpha\beta} u(x) + f_K^{\alpha\beta}(x)] \end{aligned}$$

with $f_K^{\alpha\beta}$ defined this time by

$$f_K^{\alpha\beta}(x) = f^{\alpha\beta}(x)I_{\beta \in B_1} + KI_{\beta \in B_2}$$

and $L^{\alpha\beta}u = a_{ij}^\beta D_{ij}u + b_i^\beta D_i u - c^\beta u$ for $\beta \in B_2$. This allows us to get a probabilistic representation of $v_{0,-K}$, which allows us to conclude that $|v_{0,-K} - v| \leq N_2 K^{-\gamma_2}$ in G by inspecting this time the proof of Theorem 6.1 of [21]. We thus checked the assumption (4.12).

Furthermore, it follows from Taylor's formula that in G_h^0 we have

$$|L^{\alpha\beta}u - L_h^{\alpha\beta}u| \leq Nh\|u\|_{C^3(G)}$$

for any $u \in C^3(\bar{G})$, where N is independent of α, β and h . We see that condition (4.11) is satisfied.

Finally, as is shown in [23], the family of operators H_h is $\Lambda\mathcal{L}_d$ -stable and now a simple reference to Corollary 4.3 proves our Theorem 2.1.

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